Preparing mathematics teachers: How can we do better?

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These slides are an amalgamation as well as extension of presentations that I have given in 2011–2012 at the California State University at Bakersfield, the Wisconsin IHE Conference at Wisconsin Dells, and Western Illinois University. It gives me great pleasure to acknowledge the hospitality of Javier Trigos, Kathy Stumpf, and Iraj Kalantari.

I also want to thank Richard Askey for valuable advice and for bringing the reference of Postelnicu-Greenes to my attention, and Larry Francis for many corrections.

It is now generally acknowledged that there is a crisis in mathematics teachers' content knowledge. The crisis will deepen when the Common Core Standards are implemented.

Those who rush to judgment will ask: what is wrong with our teachers? But it would be far more appropriate to ask instead:

> What have we done wrong in teacher preparation,
> and how can we do better?

This presentation will answer the first part of this question, but my suggested answer to the second part ("how can we do better?") will be much harder to implement.

What kind of mathematical education do colleges give pre-service teachers?

Answer: The college mathematics courses for pre-service teachers build on their knowledge of K–12 mathematics.

This sounds good until we take a closer look at the mathematics that is taught in K–12:

> It is the mathematics that has been embedded in K–12 math textbooks for decades.

We call this knowledge **Textbook School Mathematics (TSM)**.

TSM is to mathematics as margarine is to butter.

School math textbooks recycle this defective body of mathematical knowledge in K–12 from generation to generation. They imposed a *de facto national math curriculum* on our nation long before the advent of the Common Core Standards.

What is wrong with TSM?

To answer this question, let me set the stage with three sound bites.

1. Often it doesn't make sense.

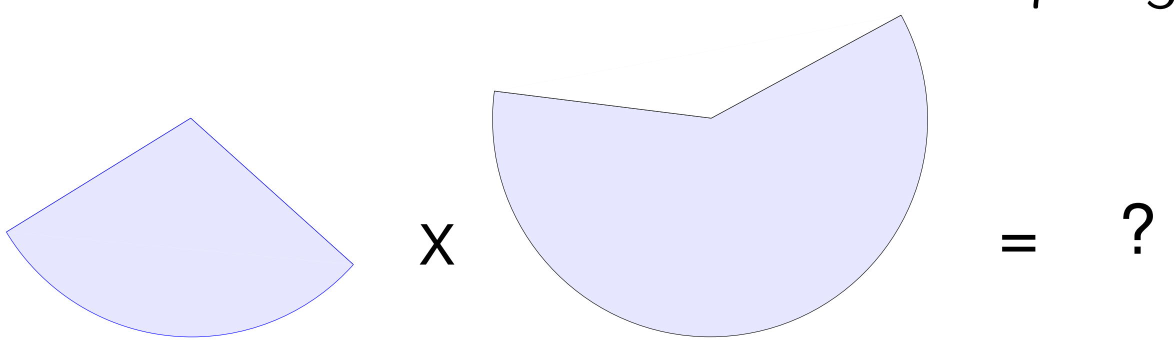
For example, students are taught to write division-with-remainder in the following way: $27 \div 4 = 6\ R\ 3$ instead of $27 = (6 \times 4) + 3$.

There are multiple reasons why this doesn't make sense. The most obvious is that by writing $69 \div 11 = 6\ R\ 3$, we conclude $27 \div 4 = 69 \div 11$, and certainly no one believes that!

In the long run, this kind of practice also contributes to the erosion of students' understanding of the equal sign.

2. It often forces students to learn mathematics only through analogies and metaphors.

For example, students are taught that a fraction is like a piece of pie. If so how to multiply two pieces of pie? $\frac{2}{7} \times \frac{3}{5} \ = ?$



Or, in fact, how to divide one piece of pie by another?

Mathematics can be motivated by analogies and metaphors, but it cannot be logically developed by analogies and metaphors.

Too often, TSM develops mathematics using **only** analogies and metaphors, devoid of logical reasoning.

This is not the kind of mathematics students should be learning.

For the multiplication of fractions, students should learn instead:

a definition of a fraction as a number (a point on the number line),

a definition of multiplying two fractions,

how *the definition of multiplication logically leads to the formula* $\frac{k}{\ell} \times \frac{m}{n} = \frac{km}{\ell n}$,

how *the definition of multiplication logically leads to the area of a rectangle with sides* $\frac{k}{\ell}$ and $\frac{m}{n}$ being $\frac{k}{\ell} \times \frac{m}{n}$.

3. It often forces students to make guesses when they should be learning how to reason.

Here is a problem in a set of national standards about *"proportional reasoning"*:

> A group of 8 people are going camping for three days and need to carry their own water. They read in a guide book that 12.5 liters are needed for a party of 5 persons for 1 day. How much water should they carry?

The idea is to assess students' ability to *reason proportionally:*

If 5 persons drink 12.5 liters a day, then one person drinks $\frac{12.5}{5} = 2.5$ liters a day, and therefore 8 persons would drink $8 \times 2.5 = 20$ liters a day. So for 3 days, 8 persons should bring $3 \times 20 = 60$ liters of water.

Well and good, except that without the hypothesis that everybody drinks (roughly) the same amount each day, none of this makes any sense.

Students should be learning how to **reason** as follows:

> *Assume that everybody drinks roughly the same amount each day.* If one person drinks ℓ liters a day, then 5 persons drink $5\,\ell$ liters a day, so $5\,\ell = 12.5$ and $\ell = 2.5$. Since 8 persons drink $8 \times \ell$ liters a day, they drink $8 \times 2.5 = 20$ liters a day, and therefore $3 \times 20 = 60$ liters in 3 days.

This is simple, learnable mathematics. If "proportional reasoning" is taught in this logical manner, do you think students would have any difficulty learning it?

Instead, TSM wants students to develop a conditioned reflex in this kind of problems so that they *immediately assume everybody drinks (roughly) the same amount each day*. The education literature goes along and promotes this conditioned reflex.

If we want students to learn to think, **we do not want them to develop this conditioned reflex,** for two reasons:

1. They should not contradict their common sense for the sake of cranking out an answer (they *know* people drink different amounts of water each day). **2.** One should *never* make up a hypothesis in a problem unless it is given explicitly.

TSM prefers proportional reasoning to be taught as an activity in which students always guess the needed assumption in order to solve a problem. This is defective mathematics. This is not learnable mathematics.

The need to find an optimal mathematical model for a given real-world situation is a serious issue. This problem could have been posed as an invitation to create models that would make it solvable.

The model that everybody drinks roughly the same amount each day for the camping situation is one possible model. However, students should not be made to believe that *this is the only realistic model.*

Here is another possible model:

> About half the people drink 3 liters a day and the other half drink 1.75 liters a day. Given 5 people, the guidebook probably played it safe and imagined that 3 out of 5 would consume $3 \times 3 = 9$ liters a day. The other 2 consume $2 \times 1.75 = 3.5$ liters a day. So these 5 persons drink $9 + 3.5 = 12.5$ liters a day.

This is also consistent with the given data, and this model would lead to a different solution of the camping problem.

Let us go beyond sound bites and consider something truly basic in school mathematics. **What does TSM have to say about solving equations, e.g.,** $\mathbf{4x - 3 = 2x}$**?**

From a typical textbook:

A **variable** *is a letter used to represent one or more numbers. An* **algebraic expression** *consists of numbers, at least one variable, and operations. An* **equation** *is a mathematical sentence formed by placing the symbol "=" between two algebraic expressions. A* **solution of the equation** *is a number so that when it is substituted for the variable in the equation, the equality is true.*

Observe that the sentence,

> An **algebraic expression** consists of numbers, at least one variable, and operations.

singles out a "variable" as an entity different from numbers.

So what is a "variable"? TSM generally plays coy and merely suggests that "it is something that varies".

Here are the usual steps for solving $4x - 3 \; = \; 2x$:

Step 1: $-2x + (4x - 3) \; = \; -2x + 2x$.

Step 2: $2x - 3 \; = \; 0$

Step 3: $(2x - 3) + 3 \; = \; 0 + 3$

Step 4: $2x \; = \; 3$

Step 5: $x \; = \; \frac{3}{2}$

How to justify Step 1 (adding $-2x$ to both sides), for example, if we don't know what a *variable* is?

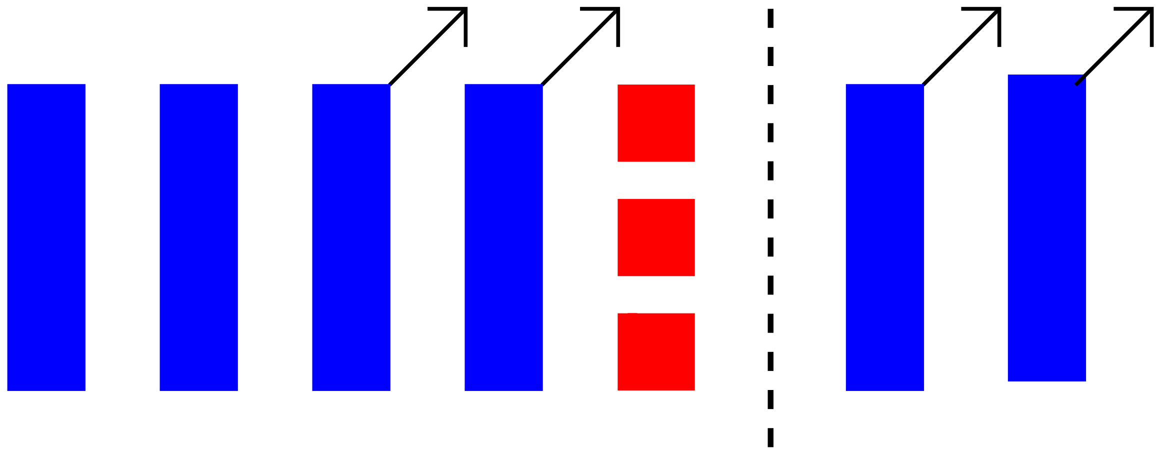
Since a variable is a different animal from a number—and TSM does not explain how to handle a variable—the equality $2x - 3 = 4x$ is a mystery. Adding the "variable" $-2x$ to both sides deepens the mystery.

There seem to be three (very plausible) strategies in TSM to deal with this mysterious move:

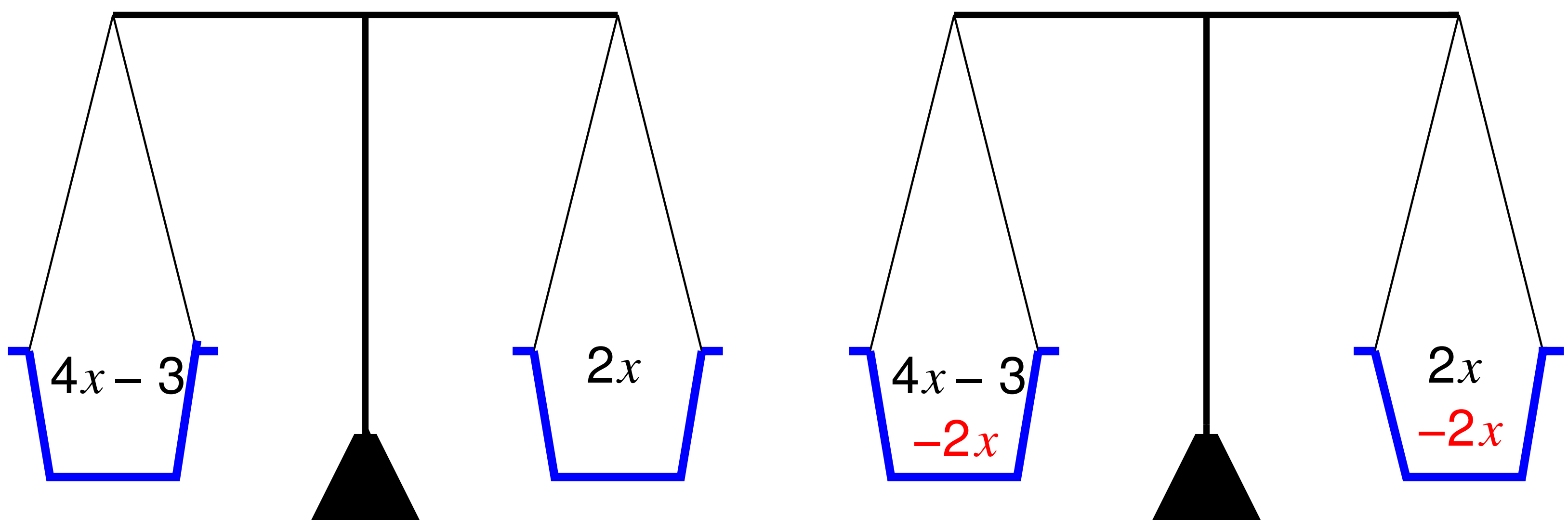
First: Invoke the principle that "Equals added to equals remain equal".

But equal as *what?* If we don't know what we are dealing with (i.e., what a "variable" is), what does it mean to say they are "equal"?

Second: Use algebra tiles to "model" this solution of $4x - 3 = 2x$. Let a blue tile model x and a red square model -1. It seems "*natural*" that if we remove two blue tiles on the left, we should also remove two blue tiles on the right.



Third: Use a balance scale to "model" this solution of $4x-3 = 2x$. It seems "*obvious*" that if we remove $2x$ (whatever it is) from both sides, the scale will stay in balance.



The other steps of solving $4x - 3 = 2x$ are justified in exactly the same way: making analogies using the intuitive meaning of "equality", algebra tiles, or balance scales.

But mathematics is supposed to *explain* why something is true by logical reasoning, not make sly suggestions about why it *might be* true by the use of analogies.

Students' **fear of variables** lives on.

In order to solve equations correctly, we should emphasize, instead, **the basic etiquette in the use of symbols:** *What a symbol stands for must be clearly stated when it is introduced.*

For example, it make no sense to say that,

> an equation is a mathematical sentence formed by placing
> the symbol $=$ between two algebraic expressions,

when the meanings of the symbols in the algebraic expressions are not clearly specified.

Let x be a real number.

An **equation in $\boldsymbol{x}$**, such as $4x - 3 = 2x$, is a *question* asking if the two numbers $4x - 3$ and $2x$ are equal, i.e., are the same number. It could be true, or it could be false.

By definition, to **solve the equation $\boldsymbol{4x - 3 = 2x}$** is to determine all the numbers x for which the equality is true.

We now show how to solve an equation. For simplicity, we once again make use of $4x - 3 = 2x$, but the principle holds in general (e.g., for quadratic equations or polynomials).

We first assume that there is a solution, i.e., there is a *number* x_o so that $4x_o - 3 = 2x_o$. *Since we are now dealing with numbers*, the previous five steps now make perfect sense.

Starting with $4x_o - 3 = 2x_o$, we get:

Step 1: $-2x_o + (4x_o - 3) \ = \ -2x_o + 2x_o$.

Step 2: $2x_o - 3 \ = \ 0$ (by use of the assoc. law for numbers)

Step 3: $(2x_o - 3) + 3 \ = \ 0 + 3$

Step 4: $2x_o \ = \ 3$ (by use of the assoc. law for numbers)

Step 5: $x_o \ = \ \frac{3}{2}$

But are we done?

No. We have not proved that $\frac{3}{2}$ *is a solution of* $4x - 3 = 2x$, only that **if** there is a solution, it must be equal to $\frac{3}{2}$.

But we can now *prove* that $\frac{3}{2}$ is a solution of $4x - 3 = 2x$ by a routine check:

$$4\left(\frac{3}{2}\right) - 3 \;=\; 2\left(\frac{3}{2}\right)$$

because both sides equal 3.

This reasoning is perfectly general.

This explanation of how equations are solved confirms that the previous Steps 1–5 are *procedurally* correct.

More importantly, it reveals that Steps 1-5 actually **make sense if only they are taught correctly**:

> Solving equations is a computation with *numbers* rather than with some mysterious quantity called a *variable*.

If we teach solving equations this way, we'd get rid of the fictitious concept of a "variable" and increase the likelihood that students learn mathematics.

According to TSM:

> *Understanding the concept of a variable is crucial to the study of algebra, and that a major problem in students' efforts to understand and do algebra results from their narrow interpretation of the term.*

The truth is, *a variable* is not a **mathematical** concept. But TSM makes it the *sine qua non* for learning algebra.

Students should be learning about the correct way to use symbols. They should learn the fact that, *in introductory algebra, every letter stands for a number.*

Recently, a teacher who was engaged in professional development in New York City said on an e-list: "I wouldn't say 90%, but a good 50% of middle school students I know who take algebra have no idea that the letters represent numbers."

This is the handiwork of TSM.

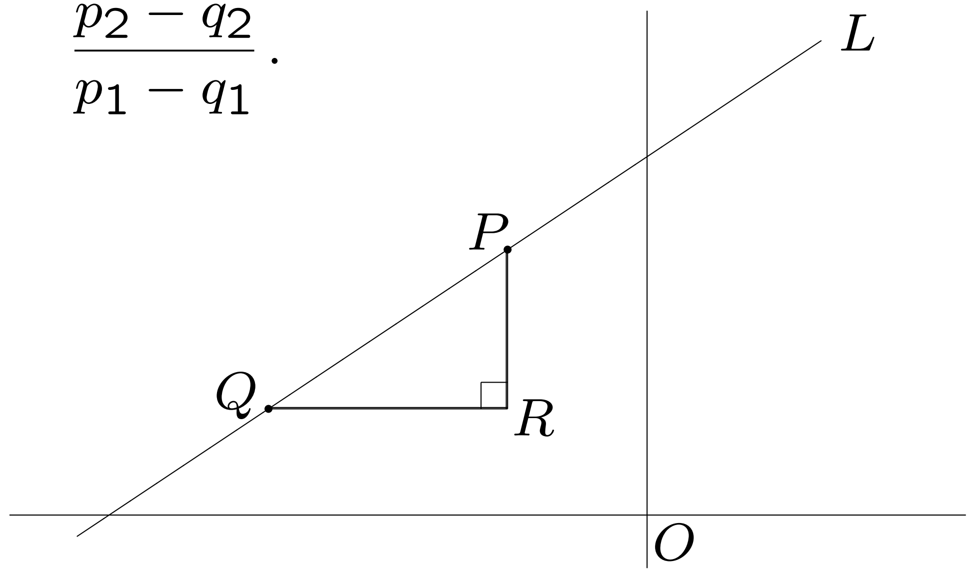
The symbolic notation as we know it today was perfected over thirty-three centuries, from the Babylonians (Pythagorean triples, circa 1700 BC) to the time of Descartes (circa 1600 AD).

The universities, being the repository of knowledge, should be jealously safeguarding the legacy of this hard-won battle and making sure that all pre-service teachers get it right.

But no such luck. Universities are indifferent, and our teachers are forced to abet TSM in the abuse of the symbolic notation.

Let us look at another substantive example: **The definition of slope.** In TSM, the definition of the **slope** of a (nonvertical) line L in the coordinate plane is the following: let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be distinct points on L. Then:

Slope of L is the ratio $\dfrac{p_2 - q_2}{p_1 - q_1}$.

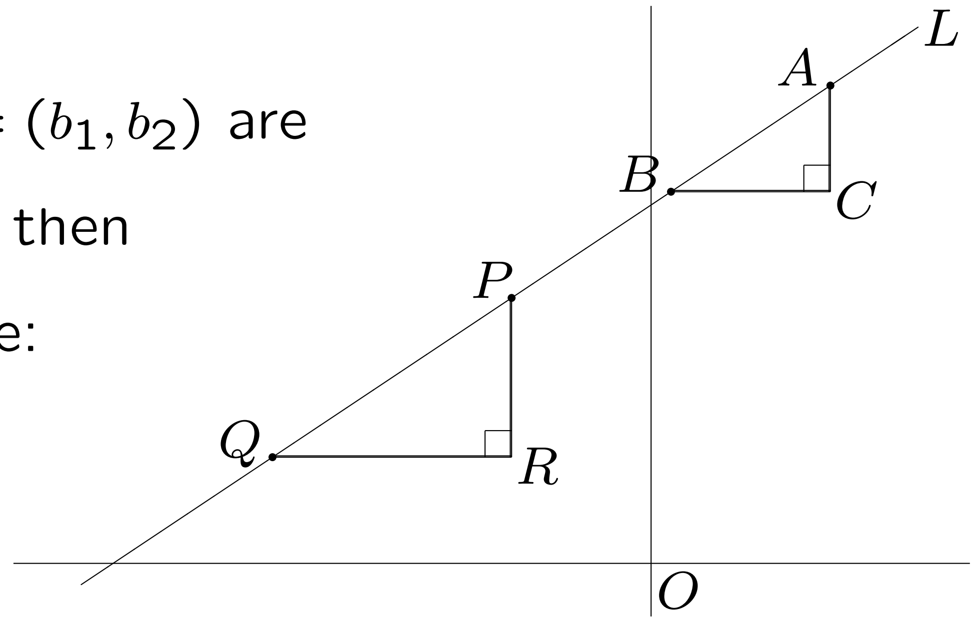


Is there anything wrong with that?

Yes, because:

If $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are two other points on L, then the slope of L would be:

$$\frac{a_2 - b_2}{a_1 - b_1}.$$



Question:

Which of these ratios should be the slope of L:

$$\frac{p_2 - q_2}{p_1 - q_1} \quad \text{or} \quad \frac{a_2 - b_2}{a_1 - b_1} \text{ ?}$$

This question must be answered if *slope* is to be a property of the line L and not of *the two chosen points on* L. It turns out $\dfrac{p_2 - q_2}{p_1 - q_1} = \dfrac{a_2 - b_2}{a_1 - b_1}$.

The fact that $\dfrac{p_2 - q_2}{p_1 - q_1} = \dfrac{a_2 - b_2}{a_1 - b_1}$ *is true* requires the concept of similar triangles: $\triangle ABC \sim \triangle PQR$.

Unfortunately, TSM wants students to *automatically assume* that these two ratios of a line must be always equal, the same way it wants students to *automatically assume* that any two people must drink the same amount of water everyday.

It is difficult to solve problems related to *slope* without the explicit knowledge that slope can be computed by *choosing any two points* that suit one's purpose.

Without this knowledge, one cannot show why the graph of $ax + by = c$ is a line, and vice versa.

Indeed, TSM does not show this.

As a result, students' difficulty with learning all aspects of the geometry of linear equations is well known, e.g., they are forced to *memorize* the four forms of a linear equation (two-point, point-slope, slope-intercept, standard) by brute force. Often without success, we may add.

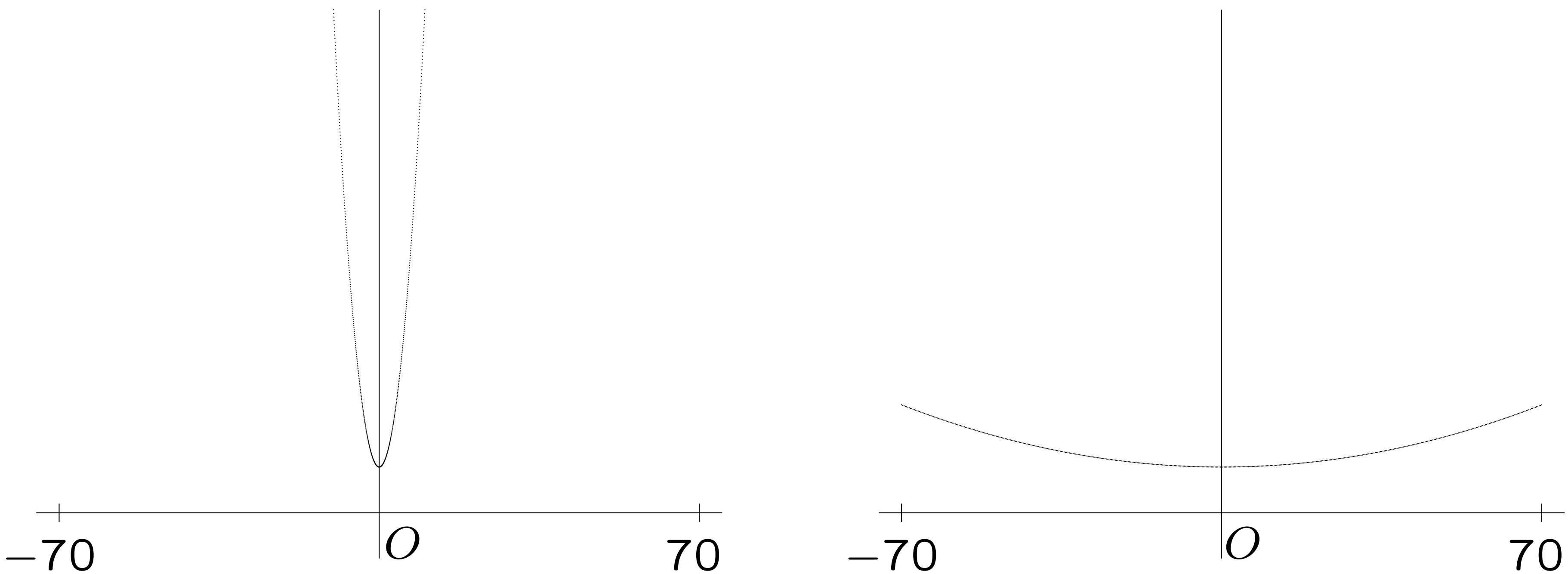
This concern about the definition of slope is not nitpicking. It is a central issue in students' learning of algebra.

A recent survey of students' understanding of straight lines in algebra, as reported by Valentina Postelnicu and Carole Greenes in the Winter 2011-2012 issue of the *NCSM Newsletter*, shows that the most difficult problems for students are those requiring the identification of the slope of a line from its graph.

The need for better teaching of the concept of slope is thus real.

TSM does not meet this need.

A final example: **What is similarity?** According to TSM, two geometric figures are **similar** if they have the same shape but not necessarily the same size. *Question:* Are the following two curves similar?



They certainly do not appear to "have the same shape", but *they are similar* in a precise sense. In fact, the left curve is the graph of

$$x^2 + 10$$

while the right curve is the graph of

$$\frac{1}{360}\, x^2 + 10,$$

and all graphs of quadratic functions are similar.

This then raises the question: How can we expect students to learn what *similarity* is so long as TSM rules the school curriculum?

The preceding examples of the flaws in TSM only begin to scratch the surface.

A better indication of the dire state of school textbooks is given in Appendix B of Chapter 3 of the National Mathematics Advisory Panel Reports of the Task Groups and Sub-Committees.

In reviewing a popular Algebra I textbook for errors, it was found that, on average, there is an error every two pages and there is a major error every 5 pages.

For a popular Algebra II textbook, there is an error every $2\frac{1}{2}$ pages, and a major error every 5 pages.

Theses error estimates are *extremely* conservative. My own estimate is that the "error density" could be 50% higher.

If I go by my own experience with textbooks, there is no question that the "error density" for K–6 textbooks is higher than this: maybe a major error every 3 pages on average.

In case you are skeptical, let me mention just a few potentially troubling spots in the K–12 curriculum so that you can read about them in standard textbooks at your leisure:

Explanation of the need for rounding (to the nearest tens, hundreds, etc.).

Proof of equivalent fractions by $\frac{m}{n} \times 1 = \frac{m}{n} \times \frac{c}{c} = \frac{mc}{nc}$.

Explanation for invert-and-multiply.

Definition of *ratio*.

Definition of *rate*; definition of *constant rate.*

Explanation of converting $\frac{m}{n}$ to a decimal by the long division of m by n.

Explanation of $(-a)(-b) = ab$ for all fractions a and b..

Explanation of $\dfrac{-m}{n} = \dfrac{m}{-n} = -\dfrac{m}{n}$ for nonzero integers m and n.

Definition of *percent.*

Is $a^{-5} = \dfrac{1}{a^5}$ a definition or a theorem?

Explanation of why the minimum of $f(x) = ax^2 + bx + c$ (for $a > 0$) is at $x = -\frac{b}{2a}$.

Explanation of why the graph of $ax + by = c$ is a line.

Explanation of why any two circles are similar.

Explanation of why $\frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{p(x)s(x)+r(x)q(x)}{q(x)s(x)}$ for *all* real numbers x.

Explanation of how to extend the domain of definition of sine and cosine from $[0, 90]$ to the real numbers.

Explanation of why $\sin(x + y) = \sin x \cos y + \cos x \sin y$ holds for *all* real numbers x and y.

In summary: Because our pre-service teachers only know TSM when they enter college, their college mathematics courses cannot build on their knowledge of K–12 mathematics and hope to produce real learning.

Colleges should intervene at this point and *systematically replace teachers' knowledge of TSM with genuine mathematics.*

Thus far, colleges have not risen to the challenge.

What about intervention *after* teachers get out of college?

In an ideal world, carefully designed and sustained *inservice* professional development (PD) can also help teachers in the field to revamp their knowledge of TSM.

But by and large, this hasn't happened either, because the awareness is not there and the needed human resources for such PD are also not there.

It therefore comes to pass that teachers teach TSM to their own students and accept TSM as the norm.

School textbook publishers happily oblige by giving teachers what they want: TSM-based materials.

This is how TSM gets recycled from generation to generation.

This is our collective fault.

Now come the Common Core Standards.

On the whole, they succeed in excluding TSM from the curriculum by being *prescriptive* about the scope and sequence of the key topics.

Implementation of the Common Core Standards requires teachers to teach **genuine mathematics**, not TSM.

Unless we can improve teachers' content knowledge, the Common Core Standards will fail disastrously.

There should be no illusion that replacing teachers' knowledge of TSM with correct mathematics is routine or easy. Not when you are talking about correcting 13 years (pre-service teachers) or 18 years (in-service teachers) of mis-education.

This task is not about teaching two or three topics better, nor is it about empowering teachers with a few mathematical or pedagogical tricks that they can use in lessons.

The task is about redoing the foundations. It is about revamping teachers' knowledge base.

It is about enabling them to think correctly, precisely, and rigorously across the board.

This cannot be accomplished by offering a one-semester "capstone course" in college, or devoting a few staff development days to content-based PD.

In greater detail:

1. *It is about defining a concept precisely before asking students to work with it;* e.g.,

number	the concept of “equal”
the concept of “less than”	fraction
product of fractions	quotient of fractions
decimal	ratio
negative number	percent
rate	constant rate
length	area
variable	expression
congruence	similarity
equation	polynomial
exponential function	logarithm

2. *It is about explaining why something is true before asking students to believe it,* such as,

Convert a fraction to a decimal by dividing the numerator by the denominator.

Invert and multiply.

$(-a)(-b) = ab$.

$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.

Solve an equation by manipulating symbols.

Define the slope of a line using only two points.

Graph $ax + by = c$ as a line.

Solve simultaneous linear equations by taking the coordinates of the intersection of the graphs.

Locate the minimum of $f(x) = ax^2 + bx + c$ $(a > 0)$ on the line $x = -\frac{b}{2a}$.

Check if c is a root of a polynomial $p(x)$ by checking whether $p(c) = 0$.

$\sin x = \cos(\frac{\pi}{2} - x)$ and $\cos x = \sin(\frac{\pi}{2} - x)$ for *all* real numbers x.

$\sin(x + y) = \sin x \cos y + \cos x \sin y$ for *all* real numbers x and y.

3. *It is about being clear whether something is being proved or being defined.* For instance, are the following statements theorems or definitions?

A fraction is a ratio.

A fraction is a division.

$\frac{k}{\ell} \times \frac{m}{n} = \frac{km}{\ell n}$ for any fractions $\frac{k}{\ell}$ and $\frac{m}{n}$.

The area of a rectangle with side lengths equal to x and y is xy for all positive x and y.

The graph of a quadratic function is a parabola.

$a^0 = 1$ for all positive a.

$a^{-1} = \frac{1}{a}$ for all positive a.

$0! = 1$.

Two lines are parallel (or coincide) if they have the same slope, and are perpendicular if the product of their slopes is -1.

Two triangles are congruent if there is a composition of rotations, translations, and reflections that maps one to the other.

For rational expressions, $\frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{p(x)s(x)+r(x)q(x)}{q(x)s(x)}$

for *all* real numbers x and y.

4. *It is about seeing—and making students see—mathematics as an organic whole rather than a random collection of facts,* e.g., they should see the progression

from addition to multiplication, and from subtraction to division,

from whole numbers to fractions, to rational numbers, to real numbers, and to complex numbers,

from arithmetic to algebra,

from reflections, rotations, and translations in middle school to high school geometry,

from length, to area, to volume.

Weeding out TSM requires sustained, painstaking work.

I have some experience in this. I have been conducting three-week summer institutes for elementary and middle school teachers every year since 2000.

Why three weeks? One week to recover from the shock of seeing the mathematics *they think they know* presented to them as correct and usable mathematics (in the classroom), and two more weeks for learning to take place.

There are also five follow-up Saturday sessions (one every two months) to discuss pedagogical ramifications. By the last session, things usually begin to sink in for those who persevere.

There is a study by Eric Hsu et al., *Seeking Big Ideas in Algebra: The Evolution of a Task*, that confirms this difficulty.

These authors asked two groups of secondary teachers to single out 5 or 6 core ideas (called "Big Ideas" in the article) that run through algebra.

The authors had in mind something like the distributive law (which can be used to explain FOIL, collecting like terms, etc.) and, to quote,

> "graphing, . . . , proportional reasoning/percents, . . . , rate of change, or solving systems of equations."

The teachers took a weekly class as well as a three-week summer session. In one group, they got started on this task *at the beginning of the spring semester,* but did not come up with a "good list of ideas" *until some time during the summer session*.

Similar result for the second group.

The teachers seemed to have no concept of logical reasoning, and were "bound to textbook use and often saw the curriculum as the contents of specific texts".

In other words, *their knowledge of mathematics consisted of nothing but TSM.*

Thus, those secondary teachers took at least half a year before they began to perceive that algebra—like all of mathematics—is developed from only a few ideas by way of logical reasoning.

Keep in mind that this says nothing about whether they got the details right. For example, do we know that they understood how equations are solved? No, we don't. Only that *they might have taken a first step toward mathematical recovery.*

Now extrapolate that to the whole K–12 math curriculum and our entire cohort of math teachers. How many years of working within this model would it take every teacher to learn how to teach mathematics correctly?

We have to break the vicious cycle of TSM.

We must smash the stranglehold of TSM on the K–12 curriculum.

Colleges must rethink their decision to teach mathematics to pre-service teachers by building on their knowledge of TSM. *There is no sense in trying to build a castle on quicksand.*

The alternatives: Talk to textbook publishers? *Forget it.* Tighten up statewide textbook adoption? *Not practical.*

Why not talk to textbook publishers?

I can tell you my firsthand experience, but better to let another person do it.

Annie Keeghan, who has worked in the educational publishing for over 20 years, has written an *exposé* of the industry's fixation on the bottom line at the expense of quality.

Teachers should not count on getting high-quality materials in the foreseeable future.

Why not count on state agencies to raise the bar? Here is an example of why not.

Recently, a state sent out an RFP for the creation of curriculum modules and PD materials in support of the implementation of the Common Core Standards.

It asked teachers, and *only* teachers, to volunteer for the review of the proposals.

It is immoral not to pay for hard work. *Worse,* by not getting knowledgeable mathematicians involved in the effort, that state in effect ensured that TSM would live on in the modules and the PD materials.

Our best hope for breaking the vicious cycle of TSM still rests with colleges. We want colleges to teach pre-service teachers the mathematics of the school curriculum *with mathematical integrity.*

Unfortunately, the preparation of math teachers in colleges, by and large, has left teachers' knowledge of TSM untouched.

All this happens for at least three reasons:

(1) Mathematicians are not aware of how TSM has poisoned school mathematics and are therefore mistaken about what teachers need.

(2) Mathematics educators—having been brought up in TSM—are not aware of the ruinous impact of TSM on teaching.

(3) Textbooks for PD are products of TSM.

We will amplify on these in the next slides.

(1) Mathematicians consider school mathematics to be too elementary and prefer to teach pre-service teachers *advanced* mathematics. They falsely believe that anyone who knows advanced mathematics will know school mathematics.

But a knowledge of rings and fields and the Cauchy Integral Theorem will not save most teachers from teaching a fraction as a piece of pie, *proportional reasoning* as a guessing game, or solving equations by appealing to a balance scale.

Most mathematicians do not know the school curriculum, much less what TSM is about.

It therefore comes to pass that when mathematicians write about the mathematical education of teachers, they talk about, e.g., reasoning and sense-making about rate problems, without realizing that teachers—victims of TSM—never learned

> what **constant rate** means,
>
> what **rate** is, or
>
> that **unit rate** makes no sense unless the rate is already known to be constant.

Mathematicians who do not know about TSM would not know how the foundational skill of division-with-remainder (DWR) has been abused (e.g., $32 \div 5 = 6\ R\ 2$), and therefore denigrated as a rote skill.

Consequently, these mathematicians do not realize that they should call elementary teachers' special attention to

> the key role DWR plays in the formulation and *proof* of the long division algorithm,*
>
> its use in converting fractions to mixed numbers,

*See, e.g., Sections 7.2–7.5 in H. Wu, *Understanding Numbers in Elementary School Mathematics*, AMS, 2011.

the use of DWR in getting the approximate location of a fraction on the number line,

the key role it plays in understanding the divisibility rules about divisions by 2, 3, 4, 5, 9, and

the key role it plays in getting the GCD of two positive integers.

Mathematicians who do not know about TSM would not know that

> it is the lack of problem-solving (including theorem-proving) in TSM that has deprived most teachers of problem-solving skills,
>
> many teachers stop trying to make sense of the mathematics they teach because the TSM they learned inherently doesn't make sense, and
>
> TSM's disregard of mathematical coherence has made many teachers confused about the difference between a definition and a theorem.

Because these mathematicians are not aware of the tremendous damage TSM has done to teachers, they ignore the *pressing need* to repair this damage, and choose instead to emphasize the importance of *research experience* in the mathematical education of teachers.

As of 2012, a *very* small percentage of our high school teachers might very well benefit from such a research experience, but for the rest, this is not what they need.

There is no sense in trying to build a castle on quicksand.

(2) Educators—*who are themselves victims of TSM*—tend to conflate TSM with mathematics.

To them, the defects of TSM are *inherent in mathematics.* One consequence is therefore that educators seek remedy primarily in pedagogy.

Another consequence is that—insofar as "content knowledge" means nothing more than "knowledge of TSM" to educators—they see content knowledge as *one of the many things a teacher needs in order to teach well*. They do not single out teachers' lack of content knowledge as the main culprit in students' non-learning of mathematics.

(3) An *overwhelming majority* of textbooks for pre- and in-service professional development are products of TSM.

This should not be surprising if we accept the fact that the authors of most of these books are either themselves products of TSM or oblivious to the very existence of TSM itself.

A first step out of this morass would be for the education community and the mathematics community to begin a dialog on TSM and the mathematics needed for teaching in schools.

The long separation between the two has damaged the needed checks and balances between mathematics and pedagogy.

At the present time, however, there is no incentive for either side to reach out to the other. Research mathematicians are preoccupied with doing research, and Schools of Education prefer to be isolated from the discipline departments.

In addition to strengthening the pre-service pipeline, we also have to help the teachers who are already in the field.

Content-based in-service PD is in shambles at the moment. Professional developers are themselves brought up by TSM; they can only teach what they know—TSM.

More serious is a fundamental conflict: A professional developer's survival instinct is to do what pleases teachers, and revamping—or replacing—teachers' content knowledge is not the best strategy for pleasing the majority of teachers.

But we must eliminate TSM.

Only when teachers realize that TSM is neither teachable not learnable will they reject textbooks that perpetuate TSM.

Only when publishers see that their books are rejected by teachers will they feel the need for improvement.

This may be the only way to break the vicious cycle.

Ultimately, there is a real incentive for eliminating TSM.

The volume **Rising Above the Gathering Storm** from the National Academy of Sciences (2007)—with an unprecedented second edition in three years in 2010—predicts the end of American leadership in science and technology in the coming decades. Its recommended action of highest priority is *"to place knowledgeable math and science teachers in the classroom"*.

Can we afford to ignore this challenge?

Realistically, TSM will be here for a while longer.

The best hope we have, currently, is to provide as many resources as possible for those teachers who reject TSM.

Unfortunately, the internet is full of traps. Despite their good intentions, many websites devoted to promoting the Common Core Standards turn out to promote TSM. I would advise extreme caution.

Let me recommend a few things about which I have firsthand knowledge. The following long files are on my homepage:

Teaching Fractions According to the Common Core Standards.
http://math.berkeley.edu/~wu/CCSS-Fractions.pdf

Teaching Geometry According to the Common Core Standards.
http://math.berkeley.edu/~wu/Progressions_Geometry.pdf

Teaching Geometry in Grade 8 and High School According to the Common Core Standards.
http://math.berkeley.edu/~wu/CCSS-Geometry.pdf

Syllabi of High School Courses According to the Common Core Standards.
http://math.berkeley.edu/~wu/Syllabi_Grades9-10.pdf

The following drafts of book manuscripts are for middle school teachers. They pre-date the Common Core Standards by several years but fully address them, especially the geometry and algebra standards of grade 8:

Pre-Algebra, http://math.berkeley.edu/~wu/Pre-Algebra.pdf

Introduction to School Algebra,
http://math.berkeley.edu/~wu/Algebrasummary.pdf

Unfortunately, the following books are not free:

Two books by T. Parker and S. Baldridge:

Elementary Mathematics for Teachers
Elementary Geometry for Teachers
http://www.singaporemath.com/College_Bookstore_s/93.htm

H. Wu, *Understanding Numbers in Elementary School Mathematics*, Amer. Math. Soc., 2011.
http://www.ams.org/bookstore-getitem/item=mbk-79

My book for middle school teachers should be out by 2015.